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Quantum Field Theory on quantized Bergman domain

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Abstract. We present an oscillator realization of discrete series representations of group $SU(2, 2)$. We give formulas for the coherent state star-product quantization of a Bergman domain D . A formulation of a (regularized) non-commutative scalar field on a quantized D is given.

Subject Classification.

1 Introduction

The fundamental role of conformal group $SO(4, 2)$ for Minkowski space-time was first stressed by Dirac, [1]. Its covering group $G = SU(2, 2)$ describes conformal properties of spinning particles, see [2], where one can found a systematic introduction to the subject. The group G and its orbits are fundamental for the twistor theory, [3]. It is also of essential importance for the ADS-CFT theory[4].

All unitary irreducible representations of the group G were classified by [6] and [5]. More general case of $SU(m, n)$ is treated in [7]. The discrete series $SU(2, 2)$ representations were used by [8] and [10] for the investigation of conformal properties of fields on Minkowski space. The highest/lowest weight of the discrete series of representations has been studied by [8] and [9].

The importance of the deformations theory for quantum systems was first stressed by [11]. The deformation method was generalized to linear Poisson structures (related to Lie algebras) in [12] and to general Poisson structures in[13]. The relation between both approaches was described in [16].

The star-product formula represents an approach going beyond deformation theory. A general star-product approach, based on coherent states on co-adjoint orbits [17], was proposed in [18], for $SU(2)$ case the star-product formula was found in [19], see also [20] for $SU(n)$ orbits (the deformed algebra can be represented as a matrix algebra). A general formula for compact Lie groups was derived in [21].

For non-compact Lie groups the situation is more complicated. The case $SU(1, 1)$ was briefly sketched in [18]. Similar approach was applied in [22] to a particular $SU(2, 2)$ orbit - the complex Minkowski space. The corresponding deformed noncommutative algebra was represented in terms of 4 bosonic oscillator pairs.

A noncommutative field theory may be defined provided the noncommutative algebra of functions, with some additional structures, is specified on a configuration space(time). For $SU(2)$ case this was done in [23] and followed

by various other papers. For the noncommutative Heisenberg group, the formulation of a noncommutative quantum field theory on a Moyal space, was given in [24]. Much more work has been done for models defined over the Euclidean deformed space-time. This culminated in [25] and [26], where a renormalizable nontrivial 4D model was found and studied. For a recent review see [27].

The noncommutative space-time model proposed by H. S. Snyder and C. N. Yang (see [28]), based on non-compact groups $SO(4, 1)$ and $SO(5, 1)$ has not been developed much further, mainly due to the success of renormalization theory approach to quantum field theory. In our opinion it could be a right time to return back to those old ideas.

The paper is organized as follows. In Section 1 we describe the Lie group $G = SU(2, 2)$ and its Lie algebra $\mathfrak{g} = su(2, 2)$. The relevant mathematical background can be found, e.g., in [29] and [30]. In Section 3 we present a simple oscillator realization (in terms of 8 bosonic oscillator pairs) of most degenerate discrete series of representations which generalizes more common (Schwinger-Jordan) oscillator realizations used in the case of compact groups. In Section 4 we construct the system of coherent states for the representation in question and we give a corresponding star-product formula for the algebra of functions on a Bergman domain D . Finally, in Section 5 we construct a quantum field theory model on the quantized Bergman domain D .

2 The group $SU(2, 2)$ and its Lie algebra

2.1 The definition of $SU(2, 2)$

The group $G = SU(2, 2)$ is the subgroup of $SL(4, \mathbf{C})$ matrices satisfying

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \Rightarrow g^\dagger \Gamma g = \Gamma, \quad \Gamma = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}. \quad (1)$$

Here all the entries a, b, c, d are 2×2 -matrices and the symbols 0 and E denote the 2×2 zero matrix and the unit matrix, respectively. Inserting this

into (1) one obtains two sets of equivalent constraints

$$a^\dagger a = E + c^\dagger c, \quad c^\dagger d = E + b^\dagger b, \quad a^\dagger b = c^\dagger d, \quad (2)$$

or,

$$a a^\dagger = E + b b^\dagger, \quad d d^\dagger = E + c c^\dagger, \quad a c^\dagger = b d^\dagger. \quad (3)$$

2.2 Maximal compact subgroup and Bergman domain

The maximal compact subgroup of $SU(2, 2)$ is $K = S(U(2) \times U(2))$ which consists of the matrices

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad k_{1,2} \in U(2), \quad \det(k_1) \det(k_2) = 1. \quad (4)$$

The corresponding Bergman domain D is a kind of Type 1 Cartan domain which defined as the group coset space:

$$D = G/K. \quad (5)$$

It can be represented as the set of all complex 2×2 matrix

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \quad (6)$$

with $Z^\dagger Z < E$. The group action on D is given by:

$$Z' = gZ = (aZ + b)(cZ + d)^{-1} \quad (7)$$

The Bergman domain D is a pseudo-convex domain where we could define Hilbert space $L^2(D, d\mu_N)$ (see (28)) of holomorphic functions with reproducing kernel $K(Z, W)$, where $Z, W \in D$. This reproducing kernel is also called Bergman kernel and it is well known that

$$K(Z, W) = \det(E - ZW^\dagger)^{-N}. \quad (8)$$

The Bergman domain D is an 8 dimensional rank 2 Hermitian symmetric space. It is also a Kähler manifold of 4 complex dimensions, with the metric given by

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(K(Z, \bar{Z})) \quad (9)$$

The topology of Bergman domain D is nontrivial. It not simply connected but has genus 4. One way to calculate the genus is by studying the corresponding complex Jordan triple. The interested reader could find more details in ([31]).

2.3 The Lie algebra \mathfrak{g} and the Haar measure

Let $\mathfrak{g} = su(2, 2)$ be the Lie algebra of G , so it is real and semisimple. It is formed by matrices satisfying

$$X^\dagger \Gamma + \Gamma X = 0 \quad (10)$$

Consider the Cartan decomposition of \mathfrak{g} (see [30], [33], [34]):

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad (11)$$

where \mathfrak{k} is the subset of all anti-hermitian matrices in \mathfrak{g}

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A^\dagger = -A, D^\dagger = -D, \text{tr}(A + D) = 0, A, D \in M_2(C) \right\}. \quad (12)$$

The set \mathfrak{k} is the Lie algebra of the maximal compact subgroup K in G . The subset \mathfrak{p} of all hermitian matrices in \mathfrak{g}

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} : B \in M_2(C) \right\} \quad (13)$$

is just a linear space and not a Lie algebra.

Let \mathfrak{a} be a maximal Abelian subalgebra in \mathfrak{p} . We may choose for \mathfrak{a} the set of all matrices of the form

$$H_\Lambda = \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} \quad (14)$$

where 0 means the 2×2 matrix with entries zeros and $\Lambda = \text{diag}(\lambda_1, \lambda_2)$ is diagonal 2×2 with λ_1, λ_2 real. The corresponding subgroup consists of all matrices of the type:

$$\delta_\Lambda = \begin{pmatrix} C & S \\ S & C \end{pmatrix}, \quad \begin{aligned} C &= \text{diag}(\text{ch}\lambda_1, \text{ch}\lambda_2), \\ S &= \text{diag}(\text{sh}\lambda_1, \text{sh}\lambda_2). \end{aligned} \quad (15)$$

Define the dual space \mathbf{a}^* spanned by the elements α_i satisfying $\alpha_i(H_\Lambda) = \lambda_i$.

Then the roots of (\mathbf{g}, \mathbf{a}) are given by

$$\pm 2\alpha_1, \pm 2\alpha_2, \pm(\alpha_1 - \alpha_2) \quad (16)$$

with multiplicities $m_{2\alpha_1} = m_{2\alpha_2} = 1$ and $m_{\alpha_1 \pm \alpha_2} = 2$.

On the root system we choose that the positive Weyl chamber given by $C^+ = \{\lambda_1, \lambda_2\}$ with $\lambda_1 > \lambda_2 > 0$. Then the positive roots are $2\alpha_1, 2\alpha_2$ and $(\alpha_1 \pm \alpha_2)$. We use Σ denote the set of all roots and Σ^+ the set of positive roots.

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \quad (17)$$

So we have

$$\rho = \alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2) + (\alpha_1 - \alpha_2) = 3\alpha_1 + \alpha_2 \quad (18)$$

Let \mathbf{a}_c be the complex extension of \mathbf{a} . Follow the same procedure as shown above we could define the complex roots $\alpha_c \in \mathbf{a}_c^*$ as

$$\alpha_c^i(\mathbf{a}_c) = \tau_i, \quad i = 1, 2. \quad (19)$$

Here τ_i are complex numbers. The formula of ρ and τ_i will be used for constructing the eigenfunction of the radial part of the invariant Laplacian. See section 5.

Now we have some physical interpretation of the Lie algebra: the first summand in formula (11) represents compact generators of *rotations*, whereas the second one represents non-compact generators *boosts*. Since the Lie algebras $su(2, 2)$ and $so(4, 2)$ are isomorphic we shall label rotations as X_{05} and x_{ab} , $a, b = 1, 2, 3, 4$, and boosts as X_{0a} and X_{a5} considering them as generators $X_{AB} = -X_{BA}$, $A, B = 0, 1, \dots, 5$, satisfying $so(4, 2)$ commutation relations

$$[X_{AB}, X_{CD}] = \eta^{AC} X_{BD} - \eta^{BC} X_{AD} + \eta^{BD} X_{AC} - \eta^{AD} X_{BC}, \quad (20)$$

with the metric tensor $\eta^{AB} = \text{diag}(+1, -1, -1, -1, -1, +1)$. Explicitly, the compact Lie algebra \mathbf{k} is spanned by 7 anti-hermitian matrices and the basis

of \mathfrak{p} is formed by 8 hermitian matrices given below:

$$\begin{aligned}
S_{05} &= \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_{j4} = \frac{i}{2} \begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix}, \quad S_{ij} = \frac{i}{2} \varepsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \\
S_{k5} &= \frac{i}{2} \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad S_{0k} = \frac{1}{2} \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \\
S_{45} &= \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_{04} = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\end{aligned} \tag{21}$$

where $i, j, k = 1, 2, 3$ and $\sigma_1, \sigma_2, \sigma_3$ are usual Pauli matrices.

The principal Cartan subalgebra

$$\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{u} \tag{22}$$

of G is spanned by three commuting generators: two noncompact X_{45} and X_{03} span \mathfrak{a} , and the additional compact one X_{12} spans \mathfrak{u} . The corresponding subgroups we shall denote as H, A and U : $H = A \times U$.

Any element of G possesses a unique Cartan decompositions

$$g = k \delta \tilde{q} = \tilde{k} \delta q, \tag{23}$$

where δ is some pure positive non-compact element of H with positive λ_1 and λ_2 given by formula (15). Further, $k = \tilde{k} u$, and $q = u \tilde{q}$ are elements of K , and u is the element of a compact subgroup U in H .

The Haar measure dg on $G = SU(2, 2)$ is, in the parametrization (23), given as

$$dg \equiv dg(\lambda, k, q) = \rho(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 d\tilde{k} d\tilde{q} du, \tag{24}$$

where $d\tilde{k}$ and $d\tilde{q}$ denotes the normalized measures on K/U , and du is the usual measure on U (see [10]). The explicit form of $\rho(\lambda_1, \lambda_2)$ is constructed from the positive roots (see Section 2.3):

$$\rho(\Lambda) = \prod_{\alpha \in \Sigma^+} |\sinh \alpha(T)|^{m_\alpha} \tag{25}$$

where m_α is the multiplicity of the positive roots. So we have:

$$\rho(\lambda_1, \lambda_2) = \text{sh}^2(\lambda_1 + \lambda_2) \text{sh}^2(\lambda_1 - \lambda_2) \text{sh}(2\lambda_1) \text{sh}(2\lambda_2). \tag{26}$$

3 Discrete series representation of $SU(2, 2)$

The group $G = SU(2, 2)$ possesses principal, supplementary and discrete series of unitary irreducible representations, see e.g., [5], [7]. The discrete series of representations is given by:

$$T_g f(Z) = [\det(CZ + D)]^{-N} f(Z'), \quad N = 4, 5, 6 \dots \quad (27)$$

where Z' is given by formula (7) and $f(Z) \in L^2(D, d\mu_N)$ with the measure

$$d\mu_N(\bar{Z}, Z) = c_N d\bar{Z} dZ \det(E - Z^\dagger Z)^{N-4}. \quad (28)$$

The normalization constant $c_N = \pi^{-4}(N-1)(N-2)^2(N-3)$ guarantees that the function $f_0(Z) \equiv 1$ has a unit norm, see [10].

We introduce a 4×2 matrix $\hat{Z} = (\hat{z}_{a\alpha})$, $a = 1, \dots, 4$, $\alpha = 1, 2$, of bosonic oscillators acting in Fock space and satisfying commutation relations

$$[\hat{z}_{a\alpha}, \hat{z}_{b\beta}^\dagger] = -\Gamma_{ab} \delta_{\alpha\beta},$$

$$[\hat{z}_{a\alpha}, \hat{z}_{b\beta}] = [\hat{z}_{a\alpha}^\dagger, \hat{z}_{b\beta}^\dagger] = 0, \quad (29)$$

where Γ is a 4×4 matrix defined in (1). It can be easily seen that for all $g \in SU(2, 2)$ these commutation relations are invariant under transformations:

$$\hat{Z} \mapsto g \hat{Z}, \quad \hat{Z}^\dagger \mapsto \hat{Z}^\dagger g^\dagger. \quad (30)$$

Since, $\Gamma = \text{diag}(+1, +1, -1, -1)$ the upper two rows in \hat{Z} corresponds to creation operators whereas the lower ones to annihilation operators:

$$\hat{Z} = \begin{pmatrix} \hat{a}^\dagger \\ \hat{b} \end{pmatrix} : [\hat{a}_{\alpha\beta}, \hat{a}_{\gamma\delta}^\dagger] = [\hat{b}_{\alpha\beta}, \hat{b}_{\gamma\delta}^\dagger] = \delta_{\alpha\beta} \delta_{\gamma\delta}, \quad \alpha, \beta, \gamma, \delta = 1, 2. \quad (31)$$

and all other commutation relations among oscillator operators vanish. The Fock space \mathcal{F} in question is generated from a normalized vacuum state $|0\rangle$, satisfying $\hat{a}_{\alpha\beta} |0\rangle = \hat{b}_{\alpha\beta} |0\rangle = 0$, by repeated actions of creation operators:

$$|m_{\alpha\beta}, n_{\alpha\beta}\rangle = \prod_{\alpha\beta} \frac{(\hat{a}_{\alpha\beta}^\dagger)^{m_{\alpha\beta}} (\hat{b}_{\alpha\beta}^\dagger)^{n_{\alpha\beta}}}{\sqrt{m_{\alpha\beta}! n_{\alpha\beta}!}} |0\rangle. \quad (32)$$

We shall use the terminology that the state $|m_{\alpha\beta}, n_{\alpha\beta}\rangle$ contains $m = \sum m_{\alpha\beta}$ particles a and $n = \sum n_{\alpha\beta}$ particles b .

The Lie algebra $su(2, 2) = so(4, 2)$ acting in Fock space can be realized in terms of oscillators. To any 4×4 matrix $X = (X_{ab})$ we assign the operator

$$\hat{X} = -\text{tr}(\hat{Z}^\dagger \Gamma X \hat{Z}) = -\hat{z}_{a\alpha}^\dagger \Gamma_{ab} X_{bc} \hat{z}_{c\alpha}, \quad (33)$$

with \hat{Z}^\dagger and \hat{Z} given in (29) and (31) in terms of oscillators. Using commutation relations for annihilation and creation operators the commutator of operators $\hat{X} = -\text{tr}(\hat{Z}^\dagger \Gamma X \hat{Z})$ and $\hat{Y} = -\text{tr}(\hat{Z}^\dagger \Gamma Y \hat{Z})$ can be easily calculated:

$$[\hat{X}, \hat{Y}] = [\text{tr}(\hat{Z}^\dagger \Gamma X \hat{Z}), \text{tr}(\hat{Z}^\dagger \Gamma Y \hat{Z})] = -\text{tr}[\hat{Z}^\dagger \Gamma (XY - YX) \hat{Z}]. \quad (34)$$

It can be easily seen that the *anti-hermitian operators*

$$\hat{X}_{AB} = -\text{tr}(\hat{Z}^\dagger \Gamma X_{AB} \hat{Z}), \quad A, B = 0, 1, \dots, 5, \quad (35)$$

with X_{AB} given in (21), satisfy in Fock space the $su(2, 2) \cong so(4, 2)$ commutation relations (20). The assignment

$$g = e^{\xi^{AB} X_{AB}} \in SU(2, 2) \Rightarrow \hat{T}(g) = e^{\xi^{AB} \hat{X}_{AB}} \quad (36)$$

then defines a *unitary* $SU(2, 2)$ *representation* in Fock space.

The adjoint action of $\hat{T}(g)$ on operators reproduces (30). In terms of a and b -oscillators in block-matrix notation this can be rewritten as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{aligned} \hat{T}(g) \hat{a}^\dagger \hat{T}^\dagger(g) &= a \hat{a}^\dagger + b \hat{b}, & \hat{T}(g) \hat{a} \hat{T}^\dagger(g) &= \hat{a} a^\dagger + \hat{b}^\dagger b^\dagger, \\ \hat{T}(g) \hat{b} \hat{T}^\dagger(g) &= d \hat{b} + c \hat{a}^\dagger, & \hat{T}(g) \hat{b}^\dagger \hat{T}^\dagger(g) &= \hat{b}^\dagger d^\dagger + \hat{a} c^\dagger. \end{aligned} \quad (37)$$

Since any $g \in SU(2, 2)$ possesses Cartan decomposition (23) we shall discuss separately rotations and special boosts given in (9). For rotations we obtain a mixing of annihilation and creation operators of a same type:

$$k = \begin{pmatrix} k' & 0 \\ 0 & k'' \end{pmatrix} : \begin{aligned} \hat{T}(k) \hat{a}^\dagger \hat{T}^\dagger(k) &= k' \hat{a}^\dagger, & \hat{T}(k) \hat{a} \hat{T}^\dagger(k) &= \hat{a} k'^\dagger, \\ \hat{T}(k) \hat{b} \hat{T}^\dagger(k) &= k'' \hat{b}, & \hat{T}(k) \hat{b}^\dagger \hat{T}^\dagger(k) &= \hat{b}^\dagger k''^\dagger. \end{aligned} \quad (38)$$

However, for special boosts from \mathbf{a} we obtain Bogolyubov transformations:

$$\delta = \begin{pmatrix} C & S \\ S & C \end{pmatrix} : \quad \begin{aligned} \hat{T}(\delta) \hat{a}^\dagger \hat{T}^\dagger(\delta) &= C \hat{a}^\dagger + S \hat{b}, & \hat{T}(\delta) \hat{a} \hat{T}^\dagger(\delta) &= \hat{a} C + \hat{b} S, \\ \hat{T}(\delta) \hat{b} \hat{T}^\dagger(\delta) &= C \hat{b} + S \hat{a}^\dagger, & \hat{T}(\delta) \hat{b}^\dagger \hat{T}^\dagger(\delta) &= \hat{b}^\dagger C + \hat{a} S, \end{aligned} \quad (39)$$

with C and S determined in (15).

Using the explicit form of matrices X_{AB} , following from (10), the action of generators can be described in terms creation and annihilation of a - and b -particles:

(i) The action of *rotation* generators results in a replacement of some a -particle by an other a -particle and by replacement of b -particle by other (ab) -particle.

(ii) The action of *boost* generators results in creation of a pair (ab) of particles or in a destruction of ab pair.

In this context it is useful to consider lowering and rising operators labeled by arbitrary 2×2 complex matrix $B = (B_{\beta\gamma})$ entering (13), that annihilate and create ab pairs:

$$\hat{T}_-^B = \hat{a}_{\alpha\beta} B_{\beta\gamma} \hat{b}_{\gamma\alpha}, \quad \hat{T}_+^B = (\hat{T}_-^B)^\dagger = \hat{a}_{\alpha\beta}^\dagger B_{\beta\gamma}^* \hat{b}_{\gamma\alpha}^\dagger. \quad (40)$$

More specifically, we can consider $B = E_{\beta\gamma}$ (the matrix with 1 in the intersection of β -th row and γ -th column and 0's otherwise). The corresponding lowering and rising operators are:

$$\hat{T}_-^{\beta\gamma} = \hat{a}_{\alpha\beta} \hat{b}_{\gamma\alpha}, \quad \hat{T}_+^{\beta\gamma} = (\hat{T}_-^{\beta\gamma})^\dagger = \hat{a}_{\alpha\beta}^\dagger \hat{b}_{\gamma\alpha}^\dagger. \quad (41)$$

In (40) and (41) the summation over α is understood. Any boost can be uniquely expressed as complex combinations of operators $\hat{T}_-^{\beta\gamma}$ and $\hat{T}_+^{\beta\gamma}$. It follows from (40) that the operator

$$\hat{N} \equiv \frac{1}{2}(\hat{N}_{\hat{b}} - \hat{N}_{\hat{b}}) = \frac{1}{2}(\hat{b}_{\alpha\beta}^\dagger \hat{b}_{\alpha\beta} - \hat{a}_{\alpha\beta}^\dagger \hat{a}_{\alpha\beta}) = \frac{1}{2}Tr(\hat{Z}^\dagger \Gamma \hat{Z}) - 2 \quad (42)$$

commutes with all generators \hat{X}_{AB} , $A, B = 0, 1, \dots, 5$.

Below, we shall restrict ourselves to most degenerate discrete series representations which are specified by the eigenvalue of the operator \hat{N} in the

representation subspace. We start to construct the representation space \mathcal{F}_N from a distinguished normalized state containing lowest number of particles:

$$\begin{aligned} |x_0\rangle &= \frac{\det(\hat{b}^\dagger)^{N-1}}{(N-1)!\sqrt{N}} |0\rangle, \quad N = 1, 2, \dots \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (-1)^n |0, 0, 0, 0; N-1-n, n, n, N-1-n\rangle. \end{aligned} \quad (43)$$

Here, $\hat{b}^\dagger = (\hat{b}_{\alpha\beta}^\dagger)$ is 2×2 matrix of b -particle creation operators, and N is a natural number that specifies the representation: $\hat{N}|x_0\rangle = (N-1)|x_0\rangle$. All other states in the representation space are obtained by the action of rising operators given in (40): such states contain besides $(2N-2)$ b -particles a finite number of ab pairs.

The maximal compact subgroup $K = S(U(2) \times U(2))$ is the stability group of the state $|x_0\rangle$. The group action for $k = \text{diag}(k', k'')$ reduces just to the phase transformation (see (38)):

$$\hat{T}(k)|x_0\rangle = \det(k''^\dagger)^{N-1} \det(k')|x_0\rangle = \det(k'')^{-N}|x_0\rangle. \quad (44)$$

The first factor comes from $\hat{b}^\dagger \mapsto \hat{b}^\dagger k''^\dagger$ (see (38)), whereas the second factor comes from $\hat{T}(k)|0\rangle = \det(k')|0\rangle$ (due to the anti-normal ordering of the compact generators containing \hat{a} and \hat{a}^\dagger).

Let us calculate the mean values of the operator $\hat{T}(g)$ in the state $|x_0\rangle$: $\omega_0(g) = \langle x_0|\hat{T}(g)|x_0\rangle$. Using Cartan decomposition $g = k\delta q$ and the action(43) of rotations we obtain:

$$\begin{aligned} \omega_0(g) &= \langle x_0|\hat{T}(g)|x_0\rangle = \langle x_0|\hat{T}(k)\hat{T}(\delta)\hat{T}(q)|x_0\rangle \\ &= \det(k'')^{-N} \det(q'')^{-N} \langle x_0|\hat{T}(\delta)|x_0\rangle. \end{aligned} \quad (45)$$

Thus it is enough to calculate the mean value for the special boost:

$$\delta = \begin{pmatrix} C & S \\ S & C \end{pmatrix} = \begin{pmatrix} E & 0 \\ T & E \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} \begin{pmatrix} E & T \\ 0 & E \end{pmatrix} = t_+ t_0 t_-. \quad (46)$$

Here $C = \text{diag}(\text{ch } \lambda_1, \text{ch } \lambda_2)$, $S = \text{diag}(\text{sh } \lambda_1, \text{sh } \lambda_2)$ and $T = \text{diag}(\text{th } \lambda_1, \text{th } \lambda_2)$. In the representation in question, the matrices t_+ and t_- are exponents of

rising and lowering operators respectively:

$$\hat{T}(t_+) = e^{\text{tr}(\hat{b}^\dagger T \hat{a}^\dagger)}, \quad \hat{T}(t_-) = e^{-\text{tr}(a T b)}.$$

Since $\hat{T}(t_-)$ contains \hat{a} its action does not affect $|x_0\rangle$, and similarly $\hat{T}(t_+)$ containing \hat{a}^\dagger does not affect $\langle x_0|$. The only non-trivial action comes from

$$\hat{T}(t_0) = e^{-\text{tr}(\hat{a} \Lambda \hat{a}^\dagger) - \text{tr}(\hat{b}^\dagger \Lambda \hat{b})}, \quad \Lambda = \ln C. \quad (47)$$

Consequently,

$$\hat{T}(t_0)|x_0\rangle = \det(C)^{-N}|x_0\rangle. \quad (48)$$

The last equality follows from the identity

$$e^{-\text{tr}(\hat{b}^\dagger \Lambda \hat{b})} \det(\hat{b}^\dagger)^N = e^{-N \text{tr}(\Lambda)} \det(\hat{b}^\dagger)^N e^{-\text{tr}(\hat{b}^\dagger \Lambda \hat{b})}$$

(which can be proven, e.g. by induction in N). From equations (44) and (47) we obtain remarkably simple results:

$$\omega_0(g) = \langle x_0 | \hat{T}(g) | x_0 \rangle = \det(k'')^{-N} \det(C)^{-N} \det(q'')^{-N} = \det(d)^{-N}. \quad (49)$$

Here $d = k'' C q''$ is the lower-right block of matrix g (see the Cartan decomposition in (45)). We recovered the results valid in the holomorphic representation (27).

4 The star product

Starting from the normalized state $|x_0\rangle \in \mathcal{F}_N$ we shall construct the Perelomov's system of coherent states for the representation in question. We choose a set of boosts of the form:

$$g_x = k \delta k^\dagger = \begin{pmatrix} k' C k'^\dagger & k' S k''^\dagger \\ k'' S k' & k'' C k''^\dagger \end{pmatrix} \equiv \begin{pmatrix} C' & \tilde{S} \\ \tilde{S}^\dagger & C'' \end{pmatrix} \in G/K, \quad (50)$$

where $k \equiv \tilde{k} = \text{diag}(k', k'')$ is an element of K with eliminated Cartan $U(1)$ factor, $C = \text{diag}(\text{ch } \lambda_1, \text{ch } \lambda_2)$ and $S = \text{diag}(\text{sh } \lambda_1, \text{sh } \lambda_2)$. Thus $x = x(k, \delta)$ depends on 8 parameters: 6 of them, hidden in k , are compact, and there

are 2 non-compact ones λ_1 and λ_2 representing special boosts. To any boost g_x , given in (50), we assign coherent state (see, [17])

$$|x\rangle = \hat{T}(g_x) |x_0\rangle = \hat{T}(k \delta k^\dagger) |x_0\rangle, \quad x = x(k, \delta). \quad (51)$$

Note: Any $x = x(k, \delta)$, can be uniquely assigned to the 2×2 complex matrix $z = k' T k''^\dagger$, $T = \text{diag}(\text{th } \lambda_1, \text{th } \lambda_2)$, forming the bounded Bergman domain $D \cong G/K$.

Let us consider operators in the representation space of the form

$$\hat{F} = \int_G dg \tilde{F}(g) \hat{T}(g), \quad (52)$$

where $\tilde{F}(g)$ is a distribution on a group G with compact support $\text{supp } \tilde{F}$. To any such operator we assign function on G/K by the prescription

$$F(x) = \langle x | \hat{F} | x \rangle = \int_G dg \tilde{F}(g) \omega(g, x), \quad (53)$$

where

$$\omega(g, x) \equiv \langle x | \hat{T}(g) | x \rangle = \omega_0(g_x^{-1} g g_x). \quad (54)$$

This equation combined with (49) offers an explicit form of $\omega(g, x)$ and is well suited for calculations.

The star-product of two functions $F(x) = \langle x | \hat{F} | x \rangle$ and $G(x) = \langle x | \hat{G} | x \rangle$ was defined in [18]:

$$\begin{aligned} (F \star G)(x) &= \langle x | \hat{F} \hat{G} | x \rangle = \int_{G \times G} dg_1 dg_2 \tilde{F}(g_1) \tilde{G}(g_2) \omega(g_1 g_2, x) \\ &= \int_G dg (\tilde{F} \circ \tilde{G})(g) \omega(g, x). \end{aligned} \quad (55)$$

In (55) the symbol $\tilde{F} \circ \tilde{G}$ denotes the convolution in the group algebra $\tilde{\mathcal{A}}_G$ of compact distributions:

$$(\tilde{F} \circ \tilde{G})(g) = \int_G dh \tilde{F}(gh^{-1}) \tilde{G}(h). \quad (56)$$

Obviously, the mapping $\tilde{F} \mapsto F$ given in (53) is a homomorphism of the group algebra $\tilde{\mathcal{A}}_G$ into the star-algebra \mathcal{A}_G^* of functions (53) on $D = G/K$. The quantized Bergman domain D_\star we identify with the noncommutative algebra of functions \mathcal{A}_G^* on D .

Note: We point out that as in the case of usual distributions, the convolution product may exist even for distribution with non-compact support provided there are satisfied specific restrictions at infinity.

It can be easily seen that

$$\text{supp}(\tilde{F} \circ \tilde{G}) \subset (\text{supp} \tilde{F})(\text{supp} \tilde{G}) \equiv \{g = g_1 g_2 \mid g_1 \in \text{supp} \tilde{F}, g_2 \in \text{supp} \tilde{G}\}.$$

Consequently, for a non-compact group there are two classes of group algebras:

(i) The first one is generated by distributions with a general compact support and the corresponding group algebra is simply the full algebra $\tilde{\mathcal{A}}_G$ defined in (55).

(ii) The second one is formed by distributions \tilde{F} with $\text{supp} \tilde{F}$ subset of a subgroup $H \subset K$, form a sub-algebra $\tilde{\mathcal{A}}_H$ of the group algebra $\tilde{\mathcal{A}}_G$.

In the second class there are two interesting extremal cases:

(a) $\tilde{\mathcal{A}}_{\{e\}}$ corresponding to the trivial subgroup $H = \{e\}$ in $G = SU(2, 2)$ ($\tilde{\mathcal{A}}_{\{e\}}$ is isomorphic to the enveloping algebra $\mathcal{U}(su(2, 2))$, see e.g., [29] or [30]).

(b) $\tilde{\mathcal{A}}_K$ corresponding to the maximal compact subgroup K in G .

The deformation quantization on Lie group co-orbits in terms of the Lie group convolution algebra $\tilde{\mathcal{A}}_{\{e\}}$ was introduced by [12]. Here we follow the related coherent state construction of the star-star on Lie group co-orbits proposed in [18].

Any distribution \tilde{F} can be given as a linear combination of finite derivatives of the group δ -function, i.e., as a linear combination of distributions

$$\tilde{F}_{A_1 B_1 \dots A_n B_n}(g) = (\mathcal{X}_{A_1 B_1} \dots \mathcal{X}_{A_n B_n} \delta)(g), \quad (57)$$

where \mathcal{X}_{AB} is the left-invariant vector field representing the generator X_{AB} . The explicit form of \mathcal{X}_{AB} in terms of the coordinates of the Bergman domain has been given by [14] and [15].

Inserting this into (53) we obtain the corresponding function from $\mathcal{A}_{\{e\}}^*$

$$F_{A_1 B_1 \dots A_n B_n}(x) = (-1)^n (\mathcal{X}_{A_n B_n} \dots \mathcal{X}_{A_1 B_1} \omega)(g, x)|_{g=e}. \quad (58)$$

Here we used the fact that the operators \mathcal{X}_{AB} are anti-hermitian differential operators with respect to the group measure dg . From (53) it follows directly that

$$\begin{aligned} & (F_{A_1 B_1 \dots A_n B_n} \star F_{C_1 D_1 \dots C_m D_m})(x) \\ &= (-1)^{n+m} (\mathcal{X}_{A_n B_n} \dots \mathcal{X}_{A_1 B_1} \mathcal{X}_{C_m D_m} \dots \mathcal{X}_{C_1 D_1} \omega)(g, x)|_{g=e}. \end{aligned} \quad (59)$$

Equations (58) and (59) describe explicitly the homomorphism $\mathcal{U}(su(2, 2)) \rightarrow \mathcal{A}_{\{e\}}^*$.

Using exponential parametrization of the group element $g = e^{\xi^{AB} X_{AB}}$ formula for the symmetrized function (58) takes simple form:

$$\begin{aligned} F_{\{A_1 B_1 \dots A_n B_n\}}(x) &= (-1)^n (\partial_{\xi_{A_1 B_1}} \dots \partial_{\xi_{A_n B_n}} \omega)(e^{\xi^{AB} X_{AB}}, x)|_{\xi=0} \\ &= (-1)^n \langle x | \hat{X}_{\{A_1 B_1} \dots \hat{X}_{A_n B_n\}} | x \rangle, \end{aligned} \quad (60)$$

where $\{\dots\}$ means symmetrization of double indexes and $\xi = 0$ means the evaluation at $\xi_{AB} = 0$ for $A, B = 0, 1, \dots, 5$. Symmetrized form a basis of the algebra in question and symmetrized elements from the center of algebra correspond to Casimir operators. In the series of representation in question all Casimir operators are given in terms of a single operator \hat{N} given in (42) which is represented by a constant function $N(x) = \langle x | \hat{N} | x \rangle = N$.

Example 1.: The function $\omega(g, x)$. Let us calculate the function $\omega(g, x) = \omega_0(g_x^{-1} g g_x)$ explicitly. Taking g and g_x as in (1) and (50) we have to calculate the product of three matrices:

$$g_x^{-1} g g_x = \begin{pmatrix} C' & -\tilde{S} \\ -\tilde{S}^\dagger & C'' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C' & \tilde{S} \\ \tilde{S}^\dagger & C'' \end{pmatrix} \equiv \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix},$$

where $d_x = C'' d C'' + C'' c \tilde{S}^\dagger - \tilde{S} c C'' - \tilde{S}^\dagger a \tilde{S}$. Using equation (49) for $\omega_0(g)$ we obtain

$$\begin{aligned} \omega(g, x) &= \det(d_x)^{-N} \\ &= \det(E - z_x^\dagger z_x)^N \det(d + c z_x^\dagger - z_x b - z_x a z_x^\dagger)^{-N} \end{aligned} \quad (61)$$

Here $z_x = C'^{-1} \tilde{S} = k' T k''^\dagger$ is 2×2 complex matrix from the Bergman domain. In this form the expression is convenient for calculations.

Example 2.: Fock space realization of the co-adjoint orbit: Our aim is to calculate the coordinates

$$\xi_{AB}(x) = \frac{1}{N} \langle x | \hat{X}_{AB} | x \rangle = \frac{1}{N} \langle x_0 | \hat{T}^\dagger(g_x) \hat{X}_{AB} \hat{T}(g_x) | x_0 \rangle, \quad (62)$$

for $A, B = 0, 1, \dots, 5$. Taking into account (37) we see that $\xi_{AB}(x) = D_{AB}^{CD}(g_x) \xi_{AB}(x_0)$, where $(D_{AB}^{CD}(g)) = Ad_g^*$ is the matrix corresponding to the group action in co-adjoint algebra. Therefore it is sufficient to evaluate the coordinates at x_0 : $\xi_{AB}(x_0) = \frac{1}{N} \langle x_0 | \hat{X}_{AB} | x_0 \rangle$. A simple calculation gives: $\xi_{45}(x_0) = 1$ with all other $\xi_{AB}(x_0) = 0$. We see that $\xi_{AB}(x)$ just forms the co-adjoint orbit generated from $\xi_{AB}(x_0)$.

Example 3. The star-product of coordinates $\xi_{AB}(x)$: We have

$$(\xi_{AB} \star \xi_{CD})(x) = \frac{1}{2N^2} \langle x | \{ \hat{X}_{AB}, \hat{X}_{CD} \} | x \rangle + \frac{1}{2N^2} \langle x | [\hat{X}_{AB}, \hat{X}_{CD}] | x \rangle,$$

where $\{ \dots \}$ denotes anti-commutator and $[\dots]$ is commutator. Therefore, the second term is

$$\frac{1}{2N^2} \langle x | [\hat{X}_{AB}, \hat{X}_{CD}] | x \rangle = \frac{1}{2N} f_{AB,CD}^{EF} \xi_{EF}(x),$$

where we used the definition of \hat{X}_{AB} and the short-hand notation for the commutator (20): $[X_{AB}, X_{CD}] = f_{AB,CD}^{EF} X_{EF}$. The first term is proportional to the symmetrized function $F_{\{AB,CD\}}$ and we can use (60):

$$\frac{1}{2N^2} \langle x | \{ \hat{X}_{AB}, \hat{X}_{CD} \} | x \rangle = (1 + A_N) \xi_{AB}(x) \xi_{CD}(x) + B_N \delta_{AB,CD},$$

where we have a usual point-wise product of functions in the first term and $\delta_{AB,CD} = (1/2)(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$ in the second one. The coefficients A_N and B_N are of order $1/N$. Last two equations give

$$(\xi_{AB} \star \xi_{CD})(x) = (1 + A_N) \xi_{AB}(x) \xi_{CD}(x) + \frac{1}{2N} f_{AB,CD}^{EF} \xi_{EF}(x) + B_N \delta_{AB,CD}. \quad (63)$$

We see that the parameter of the non-commutativity is $\lambda_N = 1/N$. For $N \rightarrow \infty$ we recover the commutative product.

Theorem 4.1 *The star product (55, 63) is associative and invariant under the transformation of $SU(2, 2)$ group.*

The proof of this theorem follows directly the definition of the star product.

5 Quantum field on a Bergman domain D

5.1 The invariant Laplacian on D

The invariant Laplacian Δ_N is defined by:

$$\Delta_N \hat{T}_g = \hat{T}_g \Delta_N \quad (64)$$

where \hat{T}_g is the representation operator given by (36). We have (see [32], [33]):

$$\begin{aligned} \Delta_N = & \text{tr}[(E - ZZ^\dagger)\bar{\partial}_Z \cdot (E - Z^\dagger Z) \cdot \partial'_Z] \\ & + \det(E - Z^\dagger Z)^{-N} \text{tr}[(E - ZZ^\dagger)\bar{\partial}_Z \cdot (E - Z^\dagger Z) \cdot \partial'_Z (\det(E - Z^\dagger Z)^N)] \end{aligned} \quad (65)$$

where $\partial_Z = (\partial_{z_{ij}})$ is the 2×2 matrix of differential operators in the variables $Z = (z_{ij}) \in D$, $\bar{\partial}_Z$ and ∂'_Z denote, respectively the complex conjugate and the transpose of the matrix operator ∂_Z . It is understood that the operators $\bar{\partial}_Z$ and ∂'_Z do not differentiate the matrices $E - ZZ^\dagger$ and $E - Z^\dagger Z$.

The Laplacian Δ_N is self-adjoint on $L^2(D, d\mu_N)$ with respect to the measure given by (28).

In what follows we consider only the radial part of the invariant Laplacian Δ_N , as this part that contains information about most interesting physical quantities, e.g., the energy levels. The radial part of the invariant Laplacian could be constructed from the roots system introduced in section 2.3, (see ([33], [34]) for more details). The radial part of the invariant Laplacian reads:

$$\Delta_N^r = \omega^{-1} \left(\sum_{i=1}^2 \frac{1}{4} L_i - \frac{N}{2} \text{th } \lambda_i \partial_{\lambda_i} \right) \omega \quad (66)$$

where

$$\omega = 2(\text{ch } 2\lambda_1 - \text{ch } 2\lambda_2) \quad (67)$$

and

$$L_i = \frac{\partial^2}{\partial \lambda_i^2} + 2\text{cth } 2\lambda_i \partial_{\lambda_i} \quad (68)$$

Let $\Phi(N, \tau_1, \tau_2)$ be the eigenfunction of Δ_N^r , we have (see[33]):

$$\Delta_N^r \Phi(N, \tau_1, \tau_2) = -\frac{1}{4}[2(N-1)^2 + \tau_1^2 + \tau_2^2]\Phi(N, \tau_1, \tau_2) \quad (69)$$

where τ_1 and τ_2 are given by (19).

The operator $-\Delta_N$ is positive: it has the continuous spectrum

$$\left[-\frac{1}{2}(N-1)^2, +\infty \right) \quad (70)$$

for arbitrary τ_i , and the discrete finite spectrum

$$(N-1)(l_1 + l_2) + l_1^2 + l_2^2 \quad (71)$$

for τ_j , $j = 1, 2$, imaginary:

$$\tau_j = -i(N-1-2l_j), \quad l_j = 0, 1, \dots, \left[\frac{N-1}{2} \right]. \quad (72)$$

Here $\left[\frac{N-1}{2} \right]$ means the integer part of $\frac{N-1}{2}$. The discrete spectrum consists of $\frac{1}{2} k(k-1)$ points, where $k = \left[\frac{N-1}{2} \right]$.

5.2 A quantum field theory model

We shall present a construction of a real scalar (Euclidean) field theory on a quantized Bergman domain. The action of the model in question reads:

$$S[\Phi] = \int d\mu_N(\xi) \left[-\frac{1}{2}\Phi \star \Delta_N \Phi + \frac{1}{2}m^2\Phi \star \Phi + V_\star(\Phi) \right], \quad (73)$$

where $\Phi = \Phi(\xi)$ is a real scalar field depending on the noncommutative coordinates ξ and $d\mu_N(\xi)$ is the measure (28) expressed in terms of ξ . We suppose that V is a polynomial of Φ bounded from below. Then we expand the scalar field Φ in terms of the eigenfunction of the invariant Laplacian:

$$\Phi = \int d\tau_1 d\tau_2 C_N(\tau_1, \tau_2) \Phi(N, \tau_1, \tau_2) + \sum_{l_1, l_2} C_{N, l_1, l_2} \Phi(N, l_1, l_2), \quad (74)$$

where we integrated over the continuous part of spectrum and summed up over the discrete part of the spectrum. The coefficients $C_N(\tau_1, \tau_2)$ and C_{N, l_1, l_2} are arbitrary real numbers.

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral over fields Φ :

$$\langle F[\Phi] \rangle = \frac{\int D\Phi e^{-S[\Phi]} F[\Phi]}{\int D\Phi e^{-S[\Phi]}}, \quad (75)$$

where $D\Phi = D_x d\Phi(x) \cong \prod_{\tau_j} dC_N(\tau_j)$ and $S[\Phi]$ denotes the corresponding action (73).

For the free field propagator we recover the quantum field theory results on a commutative Bergman domain D

$$\langle \Phi, \Phi \rangle = \frac{1}{m^2 + \frac{1}{4}[2(N-1)^2 + \tau_1^2 + \tau_2^2]} \quad (76)$$

which is valid for arbitrary τ_j :

- For the discrete part of the spectrum where $\tau_j = -i(N-1-2l_j)$, the quantum field theory is finite, it possesses a cutoff at the maximal energy level N .
- For the continuous part of the spectrum where the parameters τ_j are arbitrary, the theory is divergent. But it could be made finite after proper renormalization. This point will be studied in more detail in future publications.
- When $N = \text{finite}$, we have $\frac{1}{N}$ corrections for the vertices coming from the lowest order of the star product. The divergent behavior is similar to the semiclassical case.

6 Concluding remarks

In this paper we introduced an oscillator realization of the discrete series of $SU(2, 2)$ representations. We performed a deformation quantization over the corresponding coset space $D = SU(2, 2)/S(U(2) \times U(2))$. We presented an explicit expression of the star-product over D . Using this star product we constructed a QFT model over this noncommutative Bergman domain D_\star . This method can be applied to other $SU(m, n)$ type I Cartan domains, see

[35], where $SU(2, 1)$ is discussed in detail. Such results are of interest for both physics and mathematics. From the physical point of view, $SU(2, 2)$ is the maximal symmetry group of the (compactified) Minkowski space. It is also of interest for the ADS-CFT correspondence, as $SU(2, 2)$ is the double cover of $SO(4, 2)$ conformal group. As the $SU(2, 2)$ module, the Bergman domain D is a Kähler manifold which is important in supersymmetric Quantum field theory and string theory. In addition the Bergman domain D has nontrivial Shilov boundary, and the quantization of D could help us to understand this boundary problem in the framework of noncommutative geometry. These aspects are under study and will be discussed in a forthcoming paper.

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